STEP II, 2005, Q2 MS

Q2 (a) (i) Following the definition of $f(N)$, it is immediate that $f(12) = 12(1-1/2)(1-1/3) = 4$, and $f(180) = 180(1-1/2)(1-1/3)(1-1/5) = 48$.

(ii) The result may seem obvious but care must be taken in order to construct a complete proof. For example, $N = p_1^{\alpha_1} \dots p_k^{\alpha_k} \Rightarrow f(N) = p_1^{\alpha_1 - 1} \dots p_k^{\alpha_k - 1}(p_1 - 1) \dots (p_k - 1)$. Thus as p_i is a positive integer and $\alpha_i - 1$ is a non-negative integer for $1 \leq i \leq k$, then $f(N)$ is an integer.

(b) In each of (i),(ii),(iii), the conclusion must be made clear.

(i) As $f(3)f(9) = 2 \times 6 = 12 \neq f(27) = 18$, then the statement is false.

(ii)For any two primes p and q, $f(p)f(q) = p(1-1/p)q(1-1/q) = pq(1-1/p)(1-1/q) = f(pq)$. Hence the statement is true.

(iii) Consider $f(5) = 4$, $f(6) = 2$, $f(30) = 8 = 2 \times 4$. Then as 6 is not a prime it is clear that the statement is false.

(c)Start with $p^{m-1}(p-1) = 146410$, then without difficulty it will tbe found that $p = 11$ and $m = 5$ (not 4).