STEP II, 2005, Q1 MS

Q1 Differentiation leads to $f'(x) = 2xe^{-x^2} - 2x^3e^{-x^2}$. Since $e^{-x^2} \neq 0$ for any finite x then $f'(x) = 0 \Rightarrow x - x^3 = 0 \Rightarrow x = 0, 1, -1$

For the rest of the question, observe first that $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2)\,(*)$ within a multiplicative non - zero constant. Thus $P(x)$ can take the form $-x^4/2 + px^2 + q$ and hence substitution into $(*)$ plus equating the coefficient of x^2 and constant terms leads to a possible result for $P(x)$.

A similar argument based on setting $P(x) = \sum_{i=0}^{4} c_i x^i$ is feasible, but it involves more working and so is correspondingly more error prone.

Alternatively, one can multiply $(*)$ by e^{-x^2} and then integrate with respect to x to obtain $P(x)e^{-x^2} = \int x(x^2-a^2)(x^2-b^2)e^{-x^2}\,dx$. From $\int xe^{-x^2}\,dx = -(1/2)e^{-x^2}$ the integrals $\int x^3e^{-x^2}\,dx$ and $\int x^5e^{-x^2}\,dx$ can be evaluated by use of the integration by parts rule. It then only remains to cancel out the factor e^{-x^2} to obtain $P(x) = -x^4/2 + (a^2/2 + b^2/2 - 1)x^2 - 1 + a^2/2 + b^2/2 - a^2b^2/2$.