STEP II, 2005, Q14

14 The probability density function $\mathrm{f}(x)$ of the random variable X is given by

$$\mathrm{f}(x) = k\left[\phi(x) + \lambda \mathrm{g}(x)\right],$$

where $\phi(x)$ is the probability density function of a normal variate with mean 0 and variance 1, λ is a positive constant, and $\mathrm{g}(x)$ is a probability density function defined by

$$\mathrm{g}(x) = \begin{cases} 1/\lambda & \text{for } 0 \leqslant x \leqslant \lambda\,; \\ 0 & \text{otherwise.} \end{cases}$$

Find μ, the mean of X, in terms of λ, and prove that σ, the standard deviation of X, satisfies.

$$\sigma^2 = \frac{\lambda^4 + 4\lambda^3 + 12\lambda + 12}{12(1+\lambda)^2}\,.$$

In the case $\lambda = 2$:

(i) draw a sketch of the curve $y = \mathrm{f}(x)$;

(ii) express the cumulative distribution function of X in terms of $\Phi(x)$, the cumulative distribution function corresponding to $\phi(x)$;

(iii) evaluate $\mathrm{P}(0 < X < \mu + 2\sigma)$, given that $\Phi(\frac{2}{3} + \frac{2}{3}\sqrt{7}) = 0.9921$.