STEP II, 2004, Q8

8 Let x satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 - x^n\right)^{1/n}$$

and the condition $x = 0$ when $t = 0$.

(i) Solve the equation in the case $n = 1$ and sketch the graph of the solution for $t > 0$.

(ii) Prove that $1 - x < (1 - x^2)^{1/2}$ for $0 < x < 1$.

Use this result to sketch the graph of the solution in the case $n = 2$ for $0 < t < \frac{1}{2}\pi$, using the same axes as your previous sketch.

By setting $x = \sin y$, solve the equation in this case.

(iii) Use the result (which you need not prove)

$$(1 - x^2)^{1/2} < (1 - x^3)^{1/3} \quad \text{for} \quad 0 < x < 1,$$

to sketch, without solving the equation, the graph of the solution of the equation in the case $n = 3$ using the same axes as your previous sketches. Use your sketch to show that $x = 1$ at a value of t less than $\frac{1}{2}\pi$.