STEP II, 2004, Q7 MS

Q7 Good sketch graphs of $y = x$ and $y = 2\sin x$, in the same diagram and over the interval $[0, \pi]$, will readily show that the equation $f(x) = 0$ has exactly one root in the interval $[\pi/2, \pi]$.

• $f(3\pi/4) = \sqrt{2} - 3\pi/4$ has the same sign as $2 - 9\pi^2/16 \approx 2 - 45/8 = -29/8 < 0$. Hence as $f(\pi/2) = 2 - \pi/2 > 0$ and $f(\pi) = -\pi < 0$, then $I_1 = [\pi/2, 3\pi/4]$.

• $x = \sin 5\pi/8 \Rightarrow 2x\sqrt{1 - x^2} = \sin 3\pi/4 = 1/\sqrt{2} \Rightarrow 8x^4 - 8x^2 + 1 = 0\,(*) \Rightarrow x^2 = 1/2 + 1/(2\sqrt{2}) \approx 0.85$. $(**)$. The sign of $f(5\pi/8)$ is the same as that of $4x^2 - 25\pi^2/64 \approx 17/5 - 125/32 = -81/625 < 0$. Hence $I_2 = [\pi/2, 5\pi/8]$.

• A good approximation to $x = \sin 9\pi/16$ may also be obtained in a similar way. In fact, it will be found that $f(9\pi/16) > 0$ so that $I_3 = [9\pi/16, 5\pi/8]$.