STEP II, 2004, Q5 MS

Q5 Using the integration by parts rule it is easy to establish the results $\int_0^\pi x\sin x\,dx = \pi$ and $\int_0^\pi x\cos x\,dx = -2$.

- Write $\sin(x+t) = \sin x\cos t + \sin t\cos x$ and the result $f(t) = t + A\sin t + B\cos t$, where A and B are as defined in the question, follows immediately.
- Hence write $t + A\sin t + B\cos t = t + \int_0^\pi (x + A\sin x + B\cos x)\sin(x+t)\,dx$ $(***)$ so that as

$\int_0^\pi x\sin(x+t)\,dx = \ldots = \pi\cos t - 2\sin t$,

$\int_0^\pi \sin x\sin(x+t)\,dx = \ldots = (\pi/2)\cos t$,

$\int_0^\pi \cos x\sin(x+t)\,dx = \ldots = (\pi/2)\sin t$,

then, by considering the coefficients of $\cos t$ and $\sin t$ on both sides of $(***)$, it follows that

$A = -2 + (\pi/2)B,\ B = \pi + (\pi/2)A \Rightarrow A = -2,\ B = 0.$

Alternatively, equations for A and B can be obtained by putting $t = 0$ and $t = \pi/2$ in $(***)$.