STEP II, 2004, Q2 MS

Q2 Write $Q \equiv x^2 - \alpha|x| + 2 = [|x| - \alpha/2]^2 + 2 - \alpha^2/4$.

Thus $\alpha < 2\sqrt{2} \Rightarrow 2 - \alpha^2/4 > 0 \Rightarrow Q > 0$ for all x.

It is therefore unnecessary to consider $x > 0$ and $x < 0$ separately and even more unnecessary to use calculus methods.

- if $\alpha = 3$ then $Q \equiv (|x| - 1)(|x| - 2)$, in which case the solution set of $Q < 0$ is

$\{x : -2 < x < -1\} \cup \{x : 1 < x < 2\}$.

- The solutions in x of the equation $Q = 0$ are of the form $-x_2$, $-x_1$, x_1, x_2, where $0 < x_1 < x_2$, so that $S = 2(x_2 - x_1)$. Use of the identity $x_2 - x_1 = \sqrt{(x_2 + x_1)^2 - 4x_1x_2}$ will lead immediately to $S = 2\sqrt{\alpha^2 - 8}$. Thus $S < 2\sqrt{\alpha^2} = 2\alpha$.

- The graph of S as a function of α is that part of the hyperbola $4\alpha^2 - S^2 = 32$ which is in the first quadrant. A sketch of this graph should, therefore, leave the other quadrants empty. It should also show the curve starting at the point $(2\sqrt{2}, 0)$ and asymptotically approaching the line $S = 2\alpha$.