STEP II, 2004, Q12 MS

Q12 This question generates seven separate tasks and so it is especially important to set out responses in an orderly way.

- The sketch is unimodal and falls entirely in the first quadrant of the $x-y$ plane. In particular, $y'(0+) > 0$ and y is asymptotic to $y = 0$ as $x \to \infty$.

- For $f(x)$, the constant k is determined by $\int_0^a kxe{-}x^2\,dx = 1 \Rightarrow \ldots \Rightarrow k = 2a/\left(1-e^{-a}\right)$.
- For the mode, note first that $f'(x) = k\left[1-2ax^2\right]e^{-2ax^2}$ which is zero when $x = \sqrt{1/2a}$.

As $a < 1/2 \Rightarrow x = \sqrt{1/2a} > 1$ and $f'(x) > 0$ for any $x \in [0,1]$, then in this case $m = 1$.

On the other hand, $a \geq 1/2 \Rightarrow \sqrt{1/2a} \in [0,1]$ in which case $m = \sqrt{1/2a}$.

- To determine h, set $F(h) = 1/2$, where $F(x) = \int_0^x f(y)\,dy$. This leads to $k/2a - (k/2a)e^{-ah^2} = 1/2 \Rightarrow \ldots \Rightarrow h = \sqrt{(1/a)\ln[2/(1+e^{-a})]}$.
- $a > -\ln\left(2e^{-1/2}-1\right) \Rightarrow \ldots \Rightarrow e^{1/2} < 2/(1+e^{-a}) \Rightarrow \ldots \Rightarrow h > m$.
- $e > 1 \Rightarrow e^{-1/2} < 1 \Rightarrow 2e^{-1/2} - 1 < e^{-1/2} \Rightarrow \ln(2e^{-1/2}-1) < -1/2 \Rightarrow -\ln(2e^{-1/2}-1) > 1/2$.
- $P(X > m|X < h)P(X < h) = P(X > m \cap X < h) \Rightarrow P(X > m|X < h) = [1/2 - F(1/\sqrt{2a})]/(1/2) = 1-(k/a)\left[1-e^{-1/2}\right] = \ldots = (2e^{-1/2} - e^{-a} - 1)/(1-e^{-a})$.