STEP II, 2004, Q11 MS

Q11 (i) At full engine power, the equation of motion of A is $Pv^{1/2} - kv = m(dv/dt)$.

The result $\int 1/(Pv^{1/2} - kv)\,dv = -(2/k)\ln(P - kv^{1/2})$ + constant, together with use of the condition $v(0) = 0$, followed by some algebra will lead to $v_A = (P^2/k^2)\left(1 - e^{-kt/2m}\right)^2$ (*), where v_A is the velocity of A at time t.

To obtain v_B, the velocity of B at time t, substitute $2m$ for m and $2P$ for P in (*). Thus $v_B = (4P^2/k^2)\left(1 - e^{-kt/4m}\right)^2$

(ii) $9v_A = 4v_B \Rightarrow 9\left(1 - e^{-kt/2m}\right)^2 = 16\left(1 - e^{-kt/4m}\right)^2 \Rightarrow 9\left(1 + e^{-kt/4m}\right)^2 = 16$

$\Rightarrow \ldots \Rightarrow e^{-kt/4m} = 1/3 \Rightarrow v_A = 64P^2/81k^2$ and $v_B = 16P^2/9k^2$.

(iii) The equation of motion of A is now $m(dv_A/dt) = -kv_A$, where t is now measured from the instant at which the engine of A is switched off. Since the velocity of A at the start of this phase of the motion is $64P^2/81k^2$, then subsequently $v_A = (64P^2/81k^2)e^{-kt/m}$. By a similar argument the result $v_B = (16P^2/9k^2)e^{-kt/2m}$ will be obtained. Elimination of t will then lead to $k^2v_B^2 = 4P^2v_A$.