STEP II, 2002, Q3

3 The nth Fermat number, F_n, is defined by

$$F_n = 2^{2^n} + 1\,, \qquad n = 0,\ 1,\ 2,\ \ldots\ ,$$

where 2^{2^n} means 2 raised to the power 2^n. Calculate F_0, F_1, F_2 and F_3. Show that, for $k = 1$, $k = 2$ and $k = 3$,

$$F_0 F_1 \ldots F_{k-1} = F_k - 2\,. \qquad (*)$$

Prove, by induction, or otherwise, that $(*)$ holds for all $k \geqslant 1$. Deduce that no two Fermat numbers have a common factor greater than 1.

Hence show that there are infinitely many prime numbers.