STEP II, 2002, Q14

14 A densely populated circular island is divided into N concentric regions R_1, R_2, $\ldots$, R_N, such that the inner and outer radii of R_n are $n-1$ km and n km, respectively. The average number of road accidents that occur in any one day in R_n is $2 - n/N$, independently of the number of accidents in any other region.

Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, X, that occur in it. Show that, in the long term, the average number of recorded accidents per day will be

$$2 - \frac{1}{6}\left(1 + \frac{1}{N}\right)\left(4 - \frac{1}{N}\right).$$

[Note: $\sum\limits_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$.]

Show also that

$$\mathrm{P}(X = k) = \frac{\mathrm{e}^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N}(2n-1)(2N-n)^k\,\mathrm{e}^{n/N}.$$

Suppose now that $N = 3$ and that, on a particular day, two accidents were recorded. Show that the probability that R_2 had been selected is

$$\frac{48}{48 + 45\,\mathrm{e}^{1/3} + 25\,\mathrm{e}^{-1/3}}.$$