STEP II, 2002, Q13

13 Let $\mathrm{F}(x)$ be the cumulative distribution function of a random variable X, which satisfies $\mathrm{F}(a) = 0$ and $\mathrm{F}(b) = 1$, where $a > 0$. Let

$$\mathrm{G}(y) = \frac{\mathrm{F}(y)}{2 - \mathrm{F}(y)} .$$

Show that $\mathrm{G}(a) = 0$, $\mathrm{G}(b) = 1$ and that $\mathrm{G}'(y) \geqslant 0$. Show also that

$$\frac{1}{2} \leqslant \frac{2}{(2 - \mathrm{F}(y))^2} \leqslant 2 .$$

The random variable Y has cumulative distribution function $\mathrm{G}(y)$. Show that

$$\tfrac{1}{2}\mathrm{E}(X) \leqslant \mathrm{E}(Y) \leqslant 2\mathrm{E}(X) ,$$

and that

$$\mathrm{Var}\,(Y) \leqslant 2\,\mathrm{Var}\,(X) + \tfrac{7}{4}(\mathrm{E}(X))^2 .$$