STEP II, 2001, Q2

2 Sketch the graph of the function $[x/N]$, for $0 < x < 2N$, where the notation $[y]$ means the integer part of y. (Thus $[2.9] = 2$, $[4] = 4$.)

(i) Prove that

$$\sum_{k=1}^{2N} (-1)^{[k/N]} k = 2N - N^2.$$

(ii) Let

$$S_N = \sum_{k=1}^{2N} (-1)^{[k/N]} 2^{-k}.$$

Find S_N in terms of N and determine the limit of S_N as $N \to \infty$.