STEP II, 2001, Q14

14 Two coins A and B are tossed together. A has probability p of showing a head, and B has probability $2p$, independent of A, of showing a head, where $0 < p < \frac{1}{2}$. The random variable X takes the value 1 if A shows a head and it takes the value 0 if A shows a tail. The random variable Y takes the value 1 if B shows a head and it takes the value 0 if B shows a tail. The random variable T is defined by

$$T = \lambda X + \tfrac{1}{2}(1 - \lambda)Y.$$

Show that $\mathrm{E}(T) = p$ and find an expression for $\mathrm{Var}\,(T)$ in terms of p and λ. Show that as λ varies, the minimum of $\mathrm{Var}\,(T)$ occurs when

$$\lambda = \frac{1 - 2p}{3 - 4p}\,.$$

The two coins are tossed n times, where $n > 30$, and $\overline{T}$ is the mean value of T. Let b be a fixed positive number. Show that the maximum value of $\mathrm{P}(|\overline{T} - p| < b)$ as λ varies is approximately $2\Phi(b/s) - 1$, where Φ is the cumulative distribution function of a standard normal variate and

$$s^2 = \frac{p(1-p)(1-2p)}{(3-4p)n}\,.$$