STEP II, 2000, Q5

5 It is required to approximate a given function $\mathrm{f}(x)$, over the interval $0 \leqslant x \leqslant 1$, by the linear function λx, where λ is chosen to minimise

$$\int_0^1 \big(\mathrm{f}(x) - \lambda x\big)^2 \, \mathrm{d}x.$$

Show that

$$\lambda = 3\int_0^1 x\mathrm{f}(x) \, \mathrm{d}x.$$

The residual error, R, of this approximation process is such that

$$R^2 = \int_0^1 \big(\mathrm{f}(x) - \lambda x\big)^2 \, \mathrm{d}x.$$

Show that

$$R^2 = \int_0^1 \big(\mathrm{f}(x)\big)^2 \, \mathrm{d}x - \frac{1}{3}\lambda^2.$$

Given now that $\mathrm{f}(x) = \sin(\pi x/n)$, show that (i) for large n, $\lambda \approx \pi/n$ and (ii) $\lim_{n\to\infty} R = 0$.

Explain why, prior to any calculation, these results are to be expected.

[You may assume that, when θ is small, $\sin\theta \approx \theta - \theta^3/6$ and $\cos\theta \approx 1 - \theta^2/2$.]