STEP II, 2000, Q14

14 The random variables $X_1, X_2, \ldots, X_{2n+1}$ are independently and uniformly distributed on the interval $0 \leqslant x \leqslant 1$. The random variable Y is defined to be the median of $X_1, X_2, \ldots, X_{2n+1}$. Given that the probability density function of Y is $\mathrm{g}(y)$, where

$$\mathrm{g}(y) = \begin{cases} ky^n(1-y)^n & \text{if } 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

use the result

$$\int_0^1 y^r(1-y)^s \, dy = \frac{r!s!}{(r+s+1)!}$$

to show that $k = (2n+1)!/(n!)^2$, and evaluate $\mathrm{E}(Y)$ and $\mathrm{Var}\,(Y)$. Hence show that, for any given positive number d, the inequality

$$\mathrm{P}\left(|Y - 1/2| < d/\sqrt{n}\right) < \mathrm{P}\left(|\bar{X} - 1/2| < d/\sqrt{n}\right)$$

holds provided n is large enough, where $\bar{X}$ is the mean of $X_1, X_2, \ldots, X_{2n+1}$.

[You may assume that Y and $\bar{X}$ are normally distributed for large n.]