STEP II, 1999, Q4

4 By considering the expansions in powers of x of both sides of the identity

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

show that

$$\sum_{s=0}^{n} \binom{n}{s}^2 = \binom{2n}{n},$$

where $\binom{n}{s} = \dfrac{n!}{s!\,(n-s)!}$.

By considering similar identities, or otherwise, show also that:

(i) if n is an even integer, then

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s}^2 = (-1)^{n/2}\binom{n}{n/2};$$

(ii) $$\sum_{t=1}^{n} 2t\binom{n}{t}^2 = n\binom{2n}{n}.$$