

## Section A: Pure Mathematics

- 1** Find the three values of  $x$  for which the derivative of  $x^2e^{-x^2}$  is zero.

Given that  $a$  and  $b$  are distinct positive numbers, find a polynomial  $P(x)$  such that the derivative of  $P(x)e^{-x^2}$  is zero for  $x = 0$ ,  $x = \pm a$  and  $x = \pm b$ , but for no other values of  $x$ .

- 2** For any positive integer  $N$ , the function  $f(N)$  is defined by

$$f(N) = N \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

where  $p_1, p_2, \dots, p_k$  are the only prime numbers that are factors of  $N$ .  
Thus  $f(80) = 80\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$ .

- (a) (i) Evaluate  $f(12)$  and  $f(180)$ .  
(ii) Show that  $f(N)$  is an integer for all  $N$ .
- (b) Prove, or disprove by means of a counterexample, each of the following:  
(i)  $f(m)f(n) = f(mn)$ ;  
(ii)  $f(p)f(q) = f(pq)$  if  $p$  and  $q$  are distinct prime numbers;  
(iii)  $f(p)f(q) = f(pq)$  only if  $p$  and  $q$  are distinct prime numbers.
- (c) Find a positive integer  $m$  and a prime number  $p$  such that  $f(p^m) = 146410$ .

- 3** Give a sketch, for  $0 \leq x \leq \pi/2$ , of the curve

$$y = (\sin x - x \cos x),$$

and show that  $0 \leq y \leq 1$ .

Show that:

- (i)  $\int_0^{\pi/2} y \, dx = 2 - \frac{\pi}{2}$ ;  
(ii)  $\int_0^{\pi/2} y^2 \, dx = \frac{\pi^3}{48} - \frac{\pi}{8}$ .

Deduce that  $\pi^3 + 18\pi < 96$ .

- 4 The positive numbers  $a$ ,  $b$  and  $c$  satisfy  $bc = a^2 + 1$ . Prove that

$$\arctan\left(\frac{1}{a+b}\right) + \arctan\left(\frac{1}{a+c}\right) = \arctan\left(\frac{1}{a}\right).$$

The positive numbers  $p$ ,  $q$ ,  $r$ ,  $s$ ,  $t$ ,  $u$  and  $v$  satisfy

$$st = (p+q)^2 + 1, \quad uv = (p+r)^2 + 1, \quad qr = p^2 + 1.$$

Prove that

$$\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) + \arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right) = \arctan\left(\frac{1}{p}\right).$$

Hence show that

$$\arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{82}\right) + \arctan\left(\frac{1}{187}\right) = \arctan\left(\frac{1}{7}\right).$$

[Note that  $\arctan x$  is another notation for  $\tan^{-1} x$ .]

- 5 The angle  $A$  of triangle  $ABC$  is a right angle and the sides  $BC$ ,  $CA$  and  $AB$  are of lengths  $a$ ,  $b$  and  $c$ , respectively. Each side of the triangle is tangent to the circle  $S_1$  which is of radius  $r$ . Show that  $2r = b + c - a$ .

Each vertex of the triangle lies on the circle  $S_2$ . The ratio of the area of the region between  $S_1$  and the triangle to the area of  $S_2$  is denoted by  $R$ . Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1),$$

where  $q = \frac{b+c}{a}$ . Deduce that

$$R \leq \frac{1}{\pi(\pi - 1)}.$$

- 6 (i) Write down the general term in the expansion in powers of  $x$  of  $(1-x)^{-1}$ ,  $(1-x)^{-2}$  and  $(1-x)^{-3}$ , where  $|x| < 1$ .

Evaluate  $\sum_{n=1}^{\infty} n2^{-n}$  and  $\sum_{n=1}^{\infty} n^2 2^{-n}$ .

- (ii) Show that  $(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{x^n}{2^{2n}}$ , for  $|x| < 1$ .

Evaluate  $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n} 3^n}$  and  $\sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^n}$ .

7 The position vectors, relative to an origin  $O$ , at time  $t$  of the particles  $P$  and  $Q$  are

$$\cos t \mathbf{i} + \sin t \mathbf{j} + 0 \mathbf{k} \quad \text{and} \quad \cos\left(t + \frac{1}{4}\pi\right) \left[\frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{k}\right] + 3\sin\left(t + \frac{1}{4}\pi\right) \mathbf{j},$$

respectively, where  $0 \leq t \leq 2\pi$ .

- (i) Give a geometrical description of the motion of  $P$  and  $Q$ .
- (ii) Let  $\theta$  be the angle  $POQ$  at time  $t$  that satisfies  $0 \leq \theta \leq \pi$ . Show that

$$\cos \theta = \frac{3\sqrt{2}}{8} - \frac{1}{4} \cos\left(2t + \frac{1}{4}\pi\right).$$

- (iii) Show that the total time for which  $\theta \geq \frac{1}{4}\pi$  is  $\frac{3}{2}\pi$ .

8 For  $x \geq 0$  the curve  $C$  is defined by

$$\frac{dy}{dx} = \frac{x^3 y^2}{(1+x^2)^{5/2}}$$

with  $y = 1$  when  $x = 0$ . Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{3/2}} + \frac{1}{3}$$

and hence that for large positive  $x$

$$y \approx 3 - \frac{9}{x}.$$

Draw a sketch of  $C$ .

On a separate diagram draw a sketch of the two curves defined for  $x \geq 0$  by

$$\frac{dz}{dx} = \frac{x^3 z^3}{2(1+x^2)^{5/2}}$$

with  $z = 1$  at  $x = 0$  on one curve, and  $z = -1$  at  $x = 0$  on the other.

## Section B: Mechanics

- 9** Two particles,  $A$  and  $B$ , of masses  $m$  and  $2m$ , respectively, are placed on a line of greatest slope,  $\ell$ , of a rough inclined plane which makes an angle of  $30^\circ$  with the horizontal. The coefficient of friction between  $A$  and the plane is  $\frac{1}{6}\sqrt{3}$  and the coefficient of friction between  $B$  and the plane is  $\frac{1}{3}\sqrt{3}$ . The particles are at rest with  $B$  higher up  $\ell$  than  $A$  and are connected by a light inextensible string which is taut. A force  $P$  is applied to  $B$ .
- (i) Show that the least magnitude of  $P$  for which the two particles move upwards along  $\ell$  is  $\frac{11}{8}\sqrt{3}mg$  and give, in this case, the direction in which  $P$  acts.
- (ii) Find the least magnitude of  $P$  for which the particles do not slip downwards along  $\ell$ .
- 10** The points  $A$  and  $B$  are 180 metres apart and lie on horizontal ground. A missile is launched from  $A$  at speed of  $100 \text{ m s}^{-1}$  and at an acute angle of elevation to the line  $AB$  of  $\arcsin \frac{3}{5}$ . A time  $T$  seconds later, an anti-missile missile is launched from  $B$ , at speed of  $200 \text{ m s}^{-1}$  and at an acute angle of elevation to the line  $BA$  of  $\arcsin \frac{4}{5}$ . The motion of both missiles takes place in the vertical plane containing  $A$  and  $B$ , and the missiles collide.

Taking  $g = 10 \text{ m s}^{-2}$  and ignoring air resistance, find  $T$ .

[Note that  $\arcsin \frac{3}{5}$  is another notation for  $\sin^{-1} \frac{3}{5}$ .]

- 11** A plane is inclined at an angle  $\arctan \frac{3}{4}$  to the horizontal and a small, smooth, light pulley  $P$  is fixed to the top of the plane. A string,  $APB$ , passes over the pulley. A particle of mass  $m_1$  is attached to the string at  $A$  and rests on the inclined plane with  $AP$  parallel to a line of greatest slope in the plane. A particle of mass  $m_2$ , where  $m_2 > m_1$ , is attached to the string at  $B$  and hangs freely with  $BP$  vertical. The coefficient of friction between the particle at  $A$  and the plane is  $\frac{1}{2}$ .

The system is released from rest with the string taut. Show that the acceleration of the particles is  $\frac{m_2 - m_1}{m_2 + m_1}g$ .

At a time  $T$  after release, the string breaks. Given that the particle at  $A$  does not reach the pulley at any point in its motion, find an expression in terms of  $T$  for the time after release at which the particle at  $A$  reaches its maximum height. It is found that, regardless of when the string broke, this time is equal to the time taken by the particle at  $A$  to descend from its point of maximum height to the point at which it was released. Find the ratio  $m_1 : m_2$ .

[Note that  $\arctan \frac{3}{4}$  is another notation for  $\tan^{-1} \frac{3}{4}$ .]

## Section C: Probability and Statistics

- 12** The twins Anna and Bella share a computer and never sign their e-mails. When I e-mail them, only the twin currently online responds. The probability that it is Anna who is online is  $p$  and she answers each question I ask her truthfully with probability  $a$ , independently of all her other answers, even if a question is repeated. The probability that it is Bella who is online is  $q$ , where  $q = 1 - p$ , and she answers each question truthfully with probability  $b$ , independently of all her other answers, even if a question is repeated.
- (i) I send the twins the e-mail: ‘Toss a fair coin and answer the following question. Did the coin come down heads?’. I receive the answer ‘yes’. Show that the probability that the coin did come down heads is  $\frac{1}{2}$  if and only if  $2(ap + bq) = 1$ .
- (ii) I send the twins the e-mail: ‘Toss a fair coin and answer the following question. Did the coin come down heads?’. I receive the answer ‘yes’. I then send the e-mail: ‘Did the coin come down heads?’ and I receive the answer ‘no’. Show that the probability (taking into account these answers) that the coin did come down heads is  $\frac{1}{2}$ .
- (iii) I send the twins the e-mail: ‘Toss a fair coin and answer the following question. Did the coin come down heads?’. I receive the answer ‘yes’. I then send the e-mail: ‘Did the coin come down heads?’ and I receive the answer ‘yes’. Show that, if  $2(ap + bq) = 1$ , the probability (taking into account these answers) that the coin did come down heads is  $\frac{1}{2}$ .
- 13** The number of printing errors on any page of a large book of  $N$  pages is modelled by a Poisson variate with parameter  $\lambda$  and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of  $n$  pages (where  $n$  is much smaller than  $N$  and  $n \geq 2$ ) which contain fewer than two errors is denoted by  $Y$ . Show that  $P(Y = k) = \binom{n}{k} p^k q^{n-k}$  where  $p = (1 + \lambda)e^{-\lambda}$  and  $q = 1 - p$ .
- Show also that, if  $\lambda$  is sufficiently small,
- (i)  $q \approx \frac{1}{2}\lambda^2$ ;
- (ii) the largest value of  $n$  for which  $P(Y = n) \geq 1 - \lambda$  is approximately  $2/\lambda$ ;
- (iii)  $P(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$ .

- 14 The probability density function  $f(x)$  of the random variable  $X$  is given by

$$f(x) = k [\phi(x) + \lambda g(x)],$$

where  $\phi(x)$  is the probability density function of a normal variate with mean 0 and variance 1,  $\lambda$  is a positive constant, and  $g(x)$  is a probability density function defined by

$$g(x) = \begin{cases} 1/\lambda & \text{for } 0 \leq x \leq \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

Find  $\mu$ , the mean of  $X$ , in terms of  $\lambda$ , and prove that  $\sigma$ , the standard deviation of  $X$ , satisfies.

$$\sigma^2 = \frac{\lambda^4 + 4\lambda^3 + 12\lambda + 12}{12(1 + \lambda)^2}.$$

In the case  $\lambda = 2$ :

- (i) draw a sketch of the curve  $y = f(x)$ ;
- (ii) express the cumulative distribution function of  $X$  in terms of  $\Phi(x)$ , the cumulative distribution function corresponding to  $\phi(x)$ ;
- (iii) evaluate  $P(0 < X < \mu + 2\sigma)$ , given that  $\Phi(\frac{2}{3} + \frac{2}{3}\sqrt{7}) = 0.9921$ .