Question	Scheme	Marks	AOs
1(i)(a)	$\begin{array}{c\|cccccc} \times_{13} & 1 & 3 & 4 & 9 & 10 & 12 \\ \hline 1 & 1 & 3 & 4 & 9 & 10 & 12 \\ 3 & 3 & 9 & 12 & 1 & 4 & 10 \\ 4 & 4 & 12 & 3 & 10 & 1 & 9 \\ 9 & 9 & 1 & 10 & 3 & 12 & 4 \\ 10 & 10 & 4 & 1 & 12 & 9 & 3 \\ 12 & 12 & 10 & 9 & 4 & 3 & 1 \end{array}$ One non-identity row or columns correct	M1	1.1b
	At least four rows or four columns correct	A1	1.1b
	All correct	A1	1.1b
		(3)	
(i)(b)	$\{1,3,9\}$ (This is the only such subgroup)	B1	1.1b
		(1)	
(i)(c)	The order of the group is 6, and 4 does not divide 6, so there can be no subgroup of order 4 by Lagrange's Theorem.	B1	2.4
		(1)	
(ii)	Multiplicative identity is 1, but e.g. $1\times3\equiv_{15}3$, $3\times3\equiv_{15}9$, $6\times3\equiv_{15}3$, $9\times3\equiv_{15}12$ and $12\times3\equiv_{15}6$	M1 A1	1.1b 1.1b
	So 3 has <u>no (left) inverse</u> element and hence $(\{1,3,6,9,12\},\times_{15})$ is <u>not a group</u>.	A1	2.3
		(3)	
			(8 marks)

Notes:

(i)(a)

M1: At least one non-identity row or column must be correct

A1: Allow for four correct rows or four correct columns (including a border).

A1: Fully correct.

(i)(b)

B1: Correct subgroup found.

(i)(c)

B1: States both that the order of the group is 6 and 4 does not divide 6, and refers to Lagrange's Theorem (either by name or by quoting the result "the order of a subgroup divides the order of the group" or similar).

(ii)

M1: Identifies 1 as the identity and shows that there is an element with no inverse. E.g. may write out a partial group table and state no 1 in second row. Must give evidence -- just `no inverses' is M0.

A1: Correct multiples of 3 (all checked) or their value or correct row in table if drawn.

A1: Concludes <u>not a group</u> with reason <u>inverse axiom fails</u>. Accept `no inverses' as reason providing evidence has been given to award the M.

Question	Scheme	Marks	AOs
2(a)	$x_2 = 5\times2-1=9$ and $y_2 = 3\times2+1=7$, so $x_3 = 5\times9-7$	M1	11b
	Hence $x_3 = 38$	A1	1.1b
		(2)	
(b)	(i) $\begin{vmatrix}5-\lambda & -1\\ 3 & 1-\lambda\end{vmatrix} = 0 \Rightarrow (5-\lambda)(1-\lambda)-3\times-1=0$ $\Rightarrow \lambda^2-6\lambda+8=0 \Rightarrow \big((\lambda-2)(\lambda-4)=0\Rightarrow\big)\lambda = ..$	M1	1.1b
	$\lambda=2$ and $\lambda=4$	A1	1.1b
	(ii) $\lambda=2\Rightarrow\begin{cases}5x-y=2x\\3x+y=2y\end{cases}\Rightarrow 3x=y$ oe or $\lambda=4\Rightarrow\begin{cases}5x-y=4x\\3x+y=4y\end{cases}\Rightarrow x=y$ oe	M1	1.1b
	Either $\lambda=2\Rightarrow\mathbf{v}=k\begin{pmatrix}1\\3\end{pmatrix}$ or $\lambda=4\Rightarrow\mathbf{v}=k\begin{pmatrix}1\\1\end{pmatrix}$	A1	1.1b
	Both $\lambda=2\Rightarrow\mathbf{v}=k\begin{pmatrix}1\\3\end{pmatrix}$ and $\lambda=4\Rightarrow\mathbf{v}=k\begin{pmatrix}1\\1\end{pmatrix}$	A1	1.1b
		(5)	
(c)	$\mathbf{D}=\begin{pmatrix}2&0\\0&4\end{pmatrix}$ ft their eigenvalues either way round, OR $\mathbf{P}=\begin{pmatrix}1&1\\3&1\end{pmatrix}$ ft their eigenvectors as columns in any order.	B1ft	1.1b
	Deduces BOTH $\mathbf{D}=\begin{pmatrix}2&0\\0&4\end{pmatrix}$ AND $\mathbf{P}=\begin{pmatrix}1&1\\3&1\end{pmatrix}$ ft their eigenvalues either way round and their eigenvectors (any multiples) as columns *in the correct order corresponding to their eigenvalue*s.	B1ft	2.2a
		(2)	
(d)	$\begin{pmatrix}u_{n+1}\\v_{n+1}\end{pmatrix}=\mathbf{P}^{-1}\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix}=\mathbf{P}^{-1}\mathbf{A}\begin{pmatrix}x_n\\y_n\end{pmatrix}$ OR $\mathbf{D}\begin{pmatrix}u_n\\v_n\end{pmatrix}=\mathbf{P}^{-1}\mathbf{AP}\cdot\mathbf{P}^{-1}\begin{pmatrix}x_n\\y_n\end{pmatrix}$	M1	1.1b
	$... = \mathbf{P}^{-1}\mathbf{APP}^{-1}\begin{pmatrix}x_n\\y_n\end{pmatrix}=\mathbf{D}\begin{pmatrix}u_n\\v_n\end{pmatrix}$ * OR $... = \mathbf{P}^{-1}\mathbf{A}\begin{pmatrix}x_n\\y_n\end{pmatrix}=\mathbf{P}^{-1}\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix}=\begin{pmatrix}u_{n+1}\\v_{n+1}\end{pmatrix}$ *	A1*	2.1
		(2)	

Question	Scheme	Marks	AOs
(e)	$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix} \Rightarrow u_{n+1} = 2u_n$ and $v_{n+1} = 4v_n$ $\Rightarrow u_n = u_1 \times 2^{n-1 \text{ or } n}$ and $v_n = v_1 \times 4^{n-1 \text{ or } n}$	M1	3.1a
	$u_n = u_1 \times 2^{n-1}$ and $v_n = v_1 \times 4^{n-1}$ f.t. their eigenvalues in place of 2 and 4.	A1ft	1.1b
	So $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_n + v_n \\ 3u_n + v_n \end{pmatrix} = \begin{pmatrix} u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \\ 3u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \end{pmatrix}$ OR $\begin{cases} u_n = -\frac{1}{2}x_n + \frac{1}{2}y_n \\ v_n = \frac{3}{2}x_n - \frac{1}{2}y_n \end{cases} \Rightarrow \begin{cases} x_n = u_n + v_n = u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \\ y_n = 3u_n + v_n = 3u_1 \times 2^{n-1} + v_1 \times 4^{n-1} \end{cases}$	M1	2.1
	$x_1 = 2$, $y_1 = 1 \Rightarrow u_1 = -\frac{1}{2}$ and $v_1 = \frac{5}{2}$ found using matrix multiplication or equations.	M1	1.1b
	So $x_n = \left(-\frac{1}{2}\right)2^{n-1} + \left(\frac{5}{2}\right)4^{n-1}$ and $y_n = \left(-\frac{3}{2}\right)2^{n-1} + \left(\frac{5}{2}\right)4^{n-1}$ oe	A1	1.1b
		(5)	
			(16 marks)

Notes:

(a)

M1: Finds both x_2 and y_2 and attempts x_3 using them.

A1: Correct $x_3 = 38$

(b)(i)

M1: Forms the characteristic equation for **A** and attempts to solve (usual rules).

A1: Correct eigenvalues.

(b)(ii)

M1: Correct equations set up for at least one eigenvalue and attempt to solve.

A1: One correct eigenvector.

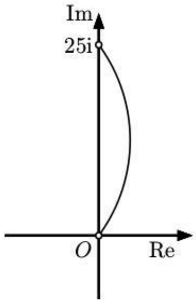
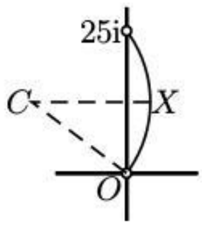
A1: Both correct.

(c)

B1ft: Either gives **D** as the diagonal matrix with their eigenvalues (either way) on diagonal, or **P** as the matrix with their eigenvectors (any multiples) as columns.

B1ft: Both **P** as the matrix with their eigenvectors (any multiples) as the columns AND **D** as the diagonal matrix with the corresponding eigenvalues on diagonal in the correct order to match the columns of **P**.

Question 2 notes continued
(d) **M1:** Changes from $\{u, v\}$ system to $\{x, y\}$ system and applies the recurrence relation to $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$. If working in reverse, for this mark both $\mathbf{D}$ and $\begin{pmatrix} u_n \\ v_n \end{pmatrix}$ must be replaced. **A1*:** Correctly derives result by introducing $\mathbf{PP}^{-1}$ before applying $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ and using the transformation again. If working in reverse it is for completing the proof correctly.
(e) **M1:** Uses their diagonal matrix to form two separate first order recurrence relations in one variable and attempts to solve each. Allow if the index is n instead of $n-1$. **A1ft:** Correct form for each, $u_1 \times \lambda_1^{n-1}$ and $v_1 \times \lambda_2^{n-1}$ for their eigenvalues. May find the closed forms for u_n and v_n using $u_1 = -\frac{1}{2}$ and $v_1 = \frac{5}{2}$ here, but the A1 needs only the correct structure. **M1:** Uses the matrix equation or solves simultaneous equations in the x_n and y_n sequences, with closed forms for u_n and v_n, to obtain forms for x_n and y_n. **M1:** Uses initial terms to find the constants -- may be awarded before the previous method mark. **A1:** Correct forms for both sequences, need not be simplified, but accept any equivalents such as $x_n = 5\times 2^{2n-3} - 2^{n-2} = \frac{1}{8}2^n\left(5\times 2^n - 2\right)$ and $y_n = 5\times 2^{2n-3} - 3\times 2^{n-2} = \frac{1}{8}2^n\left(5\times 2^n - 6\right)$

Question	Scheme	Marks	AOs
3(a)	 M1 - any arc from O to 25i in the first quadrant OR a minor arc from O to $\pm 25\text{i}$ (in any quadrant) A1 - minor arc from O to 25i in first quadrant.	M1 A1	1.1b 1.1b
		(2)	
(b)	$y = \frac{25}{2}\text{i}$ where centre is $x + y\text{i}$	B1	2.2a
	Circle passes through O so $r^2 = \left(-\frac{25}{2}\left(2+\sqrt{3}\right)\right)^2 + \left(\frac{25}{2}\right)^2$ \newline OR passes through 25i so $r^2 = \left(-\frac{25}{2}\left(2+\sqrt{3}\right)\right)^2 + \left(25 - \frac{25}{2}\right)^2$  ALT uses circle geometry to deduce angle OCX is $\frac{\pi}{12}$ or angle CXO is $11\pi/24$ and attempts to use in a correct triangle.	M1	3.1a
	$\Rightarrow r = \sqrt{25^2\left(\frac{1}{4}(4+4\sqrt{3}+3)+\frac{1}{4}\right)} = 25\sqrt{\left(2+\sqrt{3}\right)}$ ALT $r = \dfrac{12.5}{\sin\left(\frac{\pi}{12}\right)}$ or $r = \frac{25}{2}\left(2+\sqrt{3}\right) + \dfrac{25/2}{\tan\left(11\pi/24\right)}$ oe	M1	1.1b
	$\Rightarrow r = \frac{25}{2}\left(\sqrt{6}+\sqrt{2}\right)$ or $r = \text{awrt}48.296$ $(r = 48.29629131\ldots)$	A1	1.1b
		(4)	
(c)	Distance from centre of arc to first ball is $\sqrt{\left(0.5 - \left(-\frac{25}{2}\left(2+\sqrt{3}\right)\right)\right)^2 + (23.25-12.5)^2}$	M1	3.1b
	$\left(=\sqrt{47.150\ldots^2 + 10.75^2} = \sqrt{2338.7448\ldots}\right) = 48.3605\ldots$ (awrt 48.36)	A1	3.4
	So difference between distance to ball and radius of arc is $48.3605\ldots - 48.2962\ldots = 0.0642\ldots$	M1	1.1b
	$0.064 < 0.1$ (=twice radius) and so the balls will collide	A1	3.2a
		(4)	
			(10 marks)

Question 3 notes:
(a) **M1:** See scheme. **A1:** A correct sketch as per scheme.
(b) **B1:** Deduces the imaginary part of the centre is $\frac{25}{2}\text{i}$. **M1:** Uses that the circle passes through O or 25i with Pythagoras to form an equation for r^2 with x and their y. Alternatively, may use circle geometry to find the one of the angles shown with attempt to use it. **M1**: Proceeds to find r. **A1:** Correct radius, either exact or awrt to 3 d.p.
(c) **M1:** Realises the need to find the distance of the first ball from the centre of the circle, so applies Pythagoras to find this. **A1:** Correct distance. **M1:** Compares distance to radius of circle. **A1**: Correct difference, accept $(\pm)$ awrt 0.06, compared to twice the radius and conclusion that second ball will hit the first.

Question	Scheme	Marks	AOs
4	First fact gives $a-b+c=11k$, $k\in\mathbb{Z}$ OR second fact gives $a+b+c=2m+1$, $m\in\mathbb{Z}$ OR third fact gives $\left(100a+10b+c\equiv 5 \text{ (mod 9)} \Rightarrow\right)$ $a+b+c\equiv 5 \text{ (mod 9)}$ or $a+b+c=5+9p$, some $p\in\mathbb{Z}$	M1	1.1b
	Facts 1 and 2 give $a-b+c=2(m-b)+1$ is odd (oe). Facts 1 and 3 give $2b=5+9p-11k$ or $2(a+c)=5+11k+9p$ where $k\in\{0,1\}$, $p\in\{0,1,2\}$ Facts 2 and 3 give $a+b+c\equiv 5+18q$, $q\in\{0,1\}$	M1	3.1a
	Facts 1 and 2: must reach $a-b+c=11$ Facts 1 and 3: must reach $2b=12$ or 14 (both values needed) or $2(a+c)=14$ or 16 or 34 (all three needed). Facts 2 and 3: must reach $a+b+c=5$ or 23 (both values needed)	A1	2.2a
	Uses the remaining fact to find a value for b (or for $a+c$ using facts 1 and 3 initially).	M1	3.1a
	For $b=6$ or for $a+c=17$	A1	2.2a
	& Thus as $a+c=17$ so a and c must be 8 and 9, so the possibilities are 968 or 869 and no others.	A1	2.1
		(6)	
			(6 marks)

Notes:

M1: Uses any one of the facts to form an equation in a, b and c, e.g property of divisibility by 11, to form equation $a-b+c=11k$ ($k\in\mathbb{Z}$) (or $k\in\{0,1\}$ if range of values considered here), OR second fact gives $a+b+c=2k+1$ (don't accept just $a+b+c$ is odd for this mark), OR achieves $a+b+c\equiv 5 \text{ (mod 9)}$ oe from third fact (must reduce the coefficients modulo 9).

M1: Uses a second property AND combines two properties together. E.g. fact 2 gives $a+b+c=2m+1$ ($m\in\mathbb{Z}$) oe, and combining with fact one we have $a-b+c=a+b+c-2b$ is also odd, (or adds equations to deduce $2b=2m+11k+1\Rightarrow k$ odd).

Alternatives include e.g. $a+b+c-5\in\{0,9,18\}$ from fact three (since these are the only multiples of 9 possible for the ranges on a, b and c) and so $a+b+c-5=0$ or 18 (allow the M if the 0 case is missing).

A1: Correctly combines two facts with consideration of the range of values to deduce a correct equation in just a, b and c. Possibilities are $a-b+c=11$ or $a+b+c=5$ or 23 oe. (Should have both values in the latter case unless a convincing reason why it is not 5 has been given.)

M1: Uses the remaining property and attempts to solve for b. E.g. third fact gives $100a+10b+c\equiv 5 \text{ (mod 9)}$ and reduces modulo 9 to $a+b+c\equiv 5 \text{(mod 9)}$ and combines with $a-b+c=11$ to deduce $2b=9p-6$ oe so $b=\ldots$

Use of facts 1 and 3 initially will require some work of trial an elimination of possibilities here. The cases that cannot happen must be seen to be rejected.

A1: Correct deduction that $b=6$.

A1: Completes the argument and deduces correctly that the only two possible values are 968 and 869.