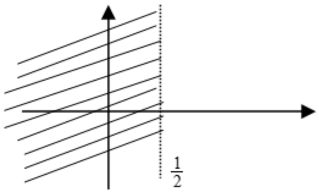
9FM0/4A: Further Pure Mathematics 02 Mark scheme

Question	Scheme		Marks	AOs
1(a)	Rearranges $w = \frac{z-1}{z}$ to either $z = f(w)$ or $z - 1 = f(w)$		M1	2.1
	$z - 1 = \frac{w}{1-w}$ or $\|z-1\| = \left\|\frac{w}{1-w}\right\|$		A1	1.1b
	As $\|z-1\| = 1 \Rightarrow \left\|\frac{w}{1-w}\right\| = 1 \Rightarrow \|w\| = \|1-w\|$ $\Rightarrow \|w\| = \|w-1\|$ *		A1*	1.1b
			(3)	
(b)		Correct line $x = ½$	M1	1.1b
		Correct shading	A1	1.1b
			(2)	
				(5 marks)

Notes:

(a)

M1: Rearranges $w = \frac{z-1}{z}$ to give $z = f(w)$ or $z - 1 = f(w)$

A1: Achieves $z - 1 = \frac{w}{1-w}$ or $|z - 1| = \left|\frac{w}{1-w}\right|$

A1*: Completes the proof, $\left|\frac{w}{1-w}\right| = 1 \Rightarrow |w| = |1-w| \Rightarrow |w| = |w-1|$*

(b)

M1: Correct line drawn $x = ½$

A1: Correct shading

Question	Scheme	Marks	AOs
2(a)	$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix} = \ldots$	M1	1.1b
	$\begin{pmatrix}6\\-6\\3\end{pmatrix} = 3\begin{pmatrix}2\\-2\\1\end{pmatrix} \therefore \begin{pmatrix}2\\-2\\1\end{pmatrix}$ is an eigenvector	A1	2.1
	Eigenvalue = 3	A1	2.2a
		(3)	
	Alternative 2a **If the candidates uses the alternative method for both parts (a) and (b) then mark them together. The first method mark in both parts can be gained simultaneously.** $\begin{vmatrix}1-\lambda&0&4\\0&5-\lambda&4\\4&4&3-\lambda\end{vmatrix} = (1-\lambda)[(5-\lambda)(3-\lambda)-16]+4[0-4(5-\lambda)] = 0$	M1	1.1b
	$(3-\lambda)(\lambda-9)(\lambda+3)=0$ $\therefore$ 3 is an eigenvalue	A1	2.2a
	$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 3\begin{pmatrix}x\\y\\z\end{pmatrix}$ leading to at least two equations $x+4z=3x,\quad 5y+4z=3y,\quad 4x+4y+3z=3z$ Attempts to solve $2z=x,\ 2z=-y,\ x+y=0$ $\therefore \begin{pmatrix}2\\-2\\1\end{pmatrix}$ is an eigenvector	A1	2.1
		(3)	
(b)	$\begin{vmatrix}-8&0&4\\0&-4&4\\4&4&-6\end{vmatrix} = -8(24-16)-0+4(0+16)$	M1	1.1b
	$= 0 \therefore 9$ is an eigenvalue	A1	2.4
	$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 9\begin{pmatrix}x\\y\\z\end{pmatrix}$ leading to at least two equations $x+4z=9x,\quad 5y+4z=9y,\quad 4x+4y+3z=9z$	M1	2.1
	Attempts to solve $z=2x,\ z=y,\ 2x+2y=3z$	M1	1.1b
	$\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$ where λ is any constant	A1	1.1b
		(5)	

	Alternative 2b $\begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix}$ $= (1-\lambda)[(5-\lambda)(3-\lambda)-16]+4[0-4(5-\lambda)]$ $= 0$	M1	1.1b
	$(3-\lambda)(\lambda-9)(\lambda+3)=0$ ∴ 9 is an eigenvalue	A1	2.4
	As main scheme	M1 M1 A1	2.1 1.1b 1.1b
(c)	$\mathbf{P} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}$	M1 A1ft	1.1b 1.1b
	$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ ∴ Eigenvalue = −3	M1	2.1
	$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, where **P** and **D** are consistent	A1	2.2a
		(4)	
			(12 marks)

Notes:

(a)

M1: Multiplies together matrix **A** and the vector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

A1: Shows that the result is a multiple of $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and concludes

A1: Deduces that 3 is an eigenvalue

(a) Alternative

If the candidates uses the alternative method for both parts (a) and (b) then mark them together. The first method mark in both parts can be gained simultaneously.

M1: Finds the determinant of **A** – λ**I** and sets = 0

A1: Factorises and shows that $\lambda = 3$ is a solution ∴ 3 is an eigenvalue

A1: Full method to deduce that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector

(b)

M1: Finds the determinant of **A** – 9**I**

A1: Shows determinant = 0 and conclusion

M1: Finds at least two equations

M1: Attempts to solve all three equations to find values for x, y and z

A1: A multiple of the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(b) Alternative

M1: Finds the determinant of **A** – λ**I** and sets = 0

A1: Factorises and shows that $\lambda = 9$ is a solution

M1: Finds at least two equations

M1: Attempts to solve all three equations to find values for *x*, *y* and *z*

A1: A multiple of the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(c)

M1 A1ft: Correct matrix **P** following through on their eigenvector from part (b)

M1: Fully correct method for finding the eigenvalue for $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

A1: Deduces the correct matrix **D**, the order of the vectors must be consistent with the order of vectors on matrix **P**

Question	Scheme		Marks	AOs
3(i)	Lottery X: ${}^{45}C_5 = 1\ 221\ 759$ and Lottery Y: ${}^{35}C_6 = 1\ 623\ 160$		M1	3.1b
	$1\ 623\ 160 > 1\ 221\ 759$ $\therefore$ lottery X as there are fewer possible outcomes, so a greater chance of winning		A1	2.4
			(2)	
(ii)	$128^{16} \equiv 1 \bmod 17$	$2^{16} = 1\text{mod}17$	B1	1.2
	$128^{16\times8+1}$	$128^{129} = 2^{16' \ 56+7}$	M1	2.1
	$128^{129} \equiv 128^{16\times8+1} \equiv 128 \bmod 17$	$128^{129} = 2^7 \text{mod}17$	A1	1.1b
	$128^{129} \equiv 9 \bmod 17$ therefore remainder is 9*		A1*	1.1b
			(4)	
(iii)(a)	$3x \equiv 2 \bmod 7$		B1	2.1
	e.g. $15x \equiv 10 \bmod 7 \Rightarrow x \equiv ..$		M1	1.1b
	$x \equiv -4 \bmod 7$ or $x \equiv 3 \bmod 7$ or $x = 7k + 3,\ k \in \mathbb{N}$		A1	1.1b
			(3)	
(iii)(b)	$x = 3, 10, 17 \ldots$		M1	3.1b
	$x = 45$		A1	1.1b
	There are 135 chairs in the room		A1	2.2a
			(3)	
				(12 marks)

Notes:
(i) **M1:** ${}^{45}C_5 = 1\ 221\ 759$ and ${}^{35}C_6 = 1\ 623\ 160$ **A1:** Compares the number of possibilities for each lotter and draws a conclusion lottery X
(ii) **B1:** Recalls Fermat's little theorem $128^{16} \equiv 1 \text{ mod } 17$ or 2^{16} º 1mod17 **M1:** Uses indices to express 128^{129} as powers of 128^{16} or powers of 2 **A1:** $128^{129} \equiv 128^{16\times 8+1} \equiv 128 \text{ mod } 17$ or 2^7mod17 **A1*:** Completes the proof $128^{129} \equiv 9 \text{ mod } 17$ therefore remainder is 9
(iii)(a) **B1:** States $3x \equiv 2 \text{ mod } 7$ **M1:** A complete method to solve the congruence equation. **A1:** $x \equiv -4 \text{ mod } 7$ or $x \equiv 3 \text{ mod } 7$
(iii)(b) **M1:** Uses their answer to part (a) to find possible values for *x* or any correct method. **A1:** $x = 45$, correct value for *x* **A1:** 135 chairs Correct answer scores full marks

Question	Scheme	Marks	AOs
4(i)	$a^5 = e$	B1	1.2
	Multiplies by $a^3 \Rightarrow a^3(a^3b) = a^3(ba^3)$ or $(a^3b)a^3 = (ba^3)a^3$	M1	2.1
	Using $a^3b = ba^3$ so that $(a^3b)a^3 = (ba^3)a^3$	M1	2.1
	Completes the proof $a^6b = ba^6$ and uses $a^6 = a \therefore ab = ba$ *	A1*	2.2a
		(4)	
(ii)(a)	A is closed under $\oplus$, the identity element is p and each element is a self-inverse	B1	2.5
	Associative law for example $(q \oplus r) \oplus s = q \oplus s = r$ but $q \oplus (r \oplus s) = q \oplus q = p$	B1	2.1
(ii)(b)	Therefore not associative and $\therefore$ the set A is not a group	B1	2.4
		(3)	
			(7 marks)

Notes:

(i)
B1: States or uses $a^5 = e$
M1: Multiplies both sides by a^3, either both on the left or right hand sides.
M1: Using $a^3b = ba^3$ so that $(a^3b)a^3 = (ba^3)a^3$
A1*: Completes the proof $a^6b = ba^6$, uses $a^6 = a$ to deduce that $ab = ba$

(ii) (a)
B1: States closure, identifies p as the identity element and finds the inverse for each element.
B1: Shows that the set A is not associative.

(ii)(b)
B1: Draws the conclusion that since it is not associative the set A is not a group.

Question	Scheme	Marks	AOs
5(a)	$\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$	M1	2.1
	$= x^n\sin x - n\left[-x^{n-1}\cos x + (n-1)\int x^{n-2}\cos x\,dx\right]$	M1 A1	3.1a 1.1b
	Using the limits $I_n = [x^n\sin x + nx^{n-1}\cos x]_0^{\frac{\pi}{2}} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\,dx$ $I_n = \left[\left(\left(\frac{\pi}{2}\right)^n \sin\left(\frac{\pi}{2}\right) + n\left(\frac{\pi}{2}\right)^{n-1}\cos\left(\frac{\pi}{2}\right)\right) - (0)\right] - n(n-1)I_{n-2}$	M1	1.1b
	$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$*	A1*	2.1
		(5)	
(b)	Area of the cross section = $(1.2 \times I_6) + \left(0.2 \times \frac{\pi}{2}\right)$ $(= 1.087\ldots)$	M1	3.1b
	$I_0 = \int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}} = 1$	B1	1.1b
	$I_6 = \left(\frac{\pi}{2}\right)^6 - 30I_4$	M1	1.1b
	$I_6 = \left(\frac{\pi}{2}\right)^6 - 30\left[\left(\frac{\pi}{2}\right)^4 - 12I_2\right]$ $I_6 = \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left[\left(\frac{\pi}{2}\right)^2 - 2I_0\right]$	M1	2.1
	$I_6 = \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$	A1	1.1b
	Volume = $10 \times$ [Area of cross section] = awrt 10.9 m^3 cso	A1	3.4
		(6)	
(c)	$10.9\text{ m}^3 = 10900$ litres $\therefore$ required time $= \frac{10900}{175}$ $= 62.3$ minutes or $175 \times 60 = 10500$ litres $= 10.5\text{ m}^3$	M1	3.1b
	62.3 minutes > 60 minutes $\therefore$ the requirement can not be met or $10.5\text{ m}^3 < 10.9\text{ m}^3$ $\therefore$ the requirement can not be met	A1	3.2a
		(2)	
			(13 marks)

Notes:
(a) **M1:** Starts integration by parts once **M1:** Integration by parts twice, to get the correct form, allow sign slips **A1:** Correct expression after using integration by parts twice **M1:** Using the limits correctly **A1*:** Completes the proof, with no errors seen.
(b) **M1:** A complete method to find the area of the cross section of the curve. **B1:** Finds $I_0 = 1$ **M1:** Finds I_6 in terms of I_4 **M1:** Finds I_6 in terms of I_0 **A1:** Correct fully numerical expression for I_6 **A1:** Correct volume = awrt 10.9 m^3 cso
(c) **M1:** A complete method to find either the rate required to fill the section within one hour or to find the volume in m^3 that can be filled in one hour **A1:** Compares and draws a conclusion

Question	Scheme	Marks	AOs
6(a)	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3\,a\cos^2\theta\sin\theta \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3\,a\sin^2\theta\cos\theta$	B1	1.1b
	$s = \int \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\,\mathrm{d}\theta$	M1	2.1
	$s = \int 3a\sqrt{\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\,\mathrm{d}\theta$	M1	1.1b
	$s = \int 3a\cos\theta\sin\theta\mathrm{d}\theta$ or $\int \frac{3}{2}a\sin2\theta\,\mathrm{d}\theta$	A1	1.1b
	$s = \int 3a\cos\theta\sin\theta\mathrm{d}\theta = \frac{3}{2}a\sin^2\theta$ or $-\frac{3}{2}a\cos^2\theta$ or $-\frac{3}{4}a\cos2\theta$	M1	2.1
	Arc length = 4 $\left[\frac{3}{2}a\sin^2\theta\right]_0^{\frac{\pi}{2}}$	M1	1.1b
	= 6a cso	A1	1.1b
		(7)	
(b)	$6a = 5 \Rightarrow a = \frac{5}{6}$	B1ft	3.4
	$A = 2\pi\int a\sin^3\theta \times 3\,a\,cos\theta\sin\theta\;\mathrm{d}\theta = 6\pi a^2 \int \sin^4\theta\cos\theta\;\mathrm{d}\theta$	M1 A1	3.1b 1.1b
	$A = \frac{6}{5}\pi a^2\sin^5\theta$	M1	1.1b
	Surface area = 2 $\left[\frac{6}{5}\pi a^2\sin^5\theta\right]_0^{\frac{\pi}{2}} = \frac{12}{5}\pi a^2\left[\sin^5\left(\frac{\pi}{2}\right) - \sin^5(0)\right]$	M1	1.1b
	Surface area = $\frac{12}{5}\pi\left(\frac{5}{6}\right)^2 = \frac{5}{3}\pi$ = awrt 5 .24	A1	3.2a
		(6)	
			(13 marks)

Notes:

(a)
M1: Correct derivatives for *x* and *y*
M1: Uses their derivatives in the correct formula for arc length
M1: Squares the derivatives and factorises out $\cos^2\theta\sin^2\theta$
A1: Using suitable identity to obtain $s = \int 3a\cos\theta\sin\theta\mathrm{d}\theta$
M1: Integrates to a multiple of $\sin^2\theta$ or $\cos^2\theta$ or $\cos2\theta$
M1: Uses limits of $\theta = \frac{\pi}{2}$ and $\theta = 0$ and multiplies by 4
A1: 6a cso

(b)

B1ft: Sets their answer to part (a) = 2 and solves to find a value for *a*

M1: Uses the correct formula for the surface area, follow through on their $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}$

A1: Correct integral

M1: Integrates to a multiple of $\sin^5\theta$

M1: Uses limits of $\theta = \frac{\pi}{2}$ and $\theta = 0$ and multiplies by 2

A1: Surface area = $\frac{5}{3}\pi$ or awrt 5.24

Question	Scheme	Marks	AOs
7(i)	Finds the characteristic equation $\lambda^2 = 6\lambda - 9$	M1	2.1
	Solves quadratic equation to give $\lambda = 3$	A1	1.1b
	Deduces the complementary function is $(A + Bn)3^n$	M1	2.2a
	Deduces the particular solution $\mu = 6\mu - 9\mu + 4 \Rightarrow \mu = 1$	B1	2.2a
	$U_n = \text{CF} + \text{PS} = (A + Bn)3^n + 1$	M1	1.1b
	Using $U_1 = 4$ and $U_2 = 7$ to find two equations in *A* and *B* $4 = (A + B)3 + 1$ and $7 = (A + 2B)9 + 1$ and solves to find at least one value of *A* or *B*	M1	2.1
	$U_n = \left(\frac{4}{3} - \frac{1}{3}n\right)3^n + 1 = (4 - n)3^{n-1} + 1$ oe so	A1	1.1b
		(7)	
(ii)	$x_1 = \frac{1}{8}[25 + 3(1 + 2 \times 1)(-1)^1] = 2$ and $x_2 = \frac{1}{8}[25 + 3(1 + 2 \times 2)(-1)^2] = 5$	B1	2.1
	Assume result is true for $n = k$ and $n = k + 1$ then for $n = k + 2$ $x_{k+2} = \ldots$	M1	2.4
	$(k + 1)x_{k+2} = (k + 2)\frac{1}{8}[25 + 3(1 + 2k)(-1)^k] -$ $\frac{1}{8}[25 + 3(1 + 2(k + 1))(-1)^{k+1}]$	A1	1.1b
	$(k + 1)x_{k+2} =$ $\frac{1}{8}[25(k + 1) + 3(-1)^{k+2}\{(k + 2)(1 + 2k)(-1)^{-2} - (3 + 2k)(-1)^{-1}\}]$ $\Rightarrow \frac{1}{8}[25(k + 1) + 3(-1)^{k+2}(2k + 5)(k + 1)]$	M1	2.1
	$x_{k+2} = \frac{1}{8}[25 + 3(2k + 5)(-1)^{k+2}]$	A1	1.1b
	If the formula is true for *n* = *k* and *n* = *k* + 1, then it is shown to be true for *n* = *k* + 2. As the result is true for *n* = 1 and *n* = 2 it is now also true for all $n \in \mathbb{Z}^+$ by mathematical induction	A1	2.2a
		(6)	
			(13 marks)

Notes:

(i)

M1: Characteristic equation $\lambda^2 = 6\lambda - 9$ or $\lambda^2 - 6\lambda + 9 = 0$

A1: Solves correct quadratic equation to achieve $\lambda = 3$

M1: Deduces the correct form of the complementary function

B1: Deduces the correct particular solution

M1: For $U_n = \text{CF} + \text{PS}$

M1: Using $U_1 = 4$ and $U_2 = 7$ to form two equations in *A* and *B* and solves to find at least one value of *A* or *B*

A1: Fully correct solution

(ii)

B1: Begins proof by induction by considering $n = 1$ and $n = 2$

M1: Assumes result is true for $n = k$ and $n = k + 1$ and uses the iterative formula to consider $n = k + 2$

A1: Correct algebraic statement for $(k + 1)x_{k+2}$ or x_{k+2}

M1: Attempts to factorise out $(k + 1)$

A1: Correct statement for x_{k+2} in required form

A1: Completes and deduces the inductive argument. All previous marks achieved.