

# 3 AS Further Mathematics

Students sitting an AS Level Further Mathematics paper may also require those formulae listed for A Level Mathematics in Section 2.

## Pure Mathematics

### Summations

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

### Matrix transformations

Anticlockwise rotation through  $\theta$  about  $O$ :  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Reflection in the line  $y = (\tan \theta)x$ :  $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

### Area of a sector

$$A = \frac{1}{2} \int r^2 d\theta \quad (\text{polar coordinates})$$

### Complex numbers

$$\{r(\cos \theta + i \sin \theta)\}^n = r^n (\cos n\theta + i \sin n\theta)$$

The roots of  $z^n = 1$  are given by  $z = e^{\frac{2\pi ki}{n}}$ , for  $k = 0, 1, 2, \dots, n-1$

## Maclaurin's and Taylor's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 \leq x \leq 1)$$

## Vectors

$$\text{Vector product: } \mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

If  $A$  is the point with position vector  $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$  and the direction vector  $\mathbf{b}$  is given by  $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ , then the straight line through  $A$  with direction vector  $\mathbf{b}$  has cartesian equation

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} (= \lambda)$$

The plane through  $A$  with normal vector  $\mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}$  has cartesian equation

$$n_1 x + n_2 y + n_3 z + d = 0 \quad \text{where } d = -\mathbf{a} \cdot \mathbf{n}$$

The plane through non-collinear points  $A$ ,  $B$  and  $C$  has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

The plane through the point with position vector  $\mathbf{a}$  and parallel to  $\mathbf{b}$  and  $\mathbf{c}$  has equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

The perpendicular distance of  $(\alpha, \beta, \gamma)$  from  $n_1 x + n_2 y + n_3 z + d = 0$  is  $\frac{|n_1 \alpha + n_2 \beta + n_3 \gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$ .

## Hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\operatorname{arcosh} x = \ln \{x + \sqrt{x^2 - 1}\} \quad (x \geq 1)$$

$$\operatorname{arsinh} x = \ln \{x + \sqrt{x^2 + 1}\}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \quad (|x| < 1)$$

## Differentiation

$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

**Integration (+ constant;  $a > 0$  where relevant)**

$f(x)$	$\int f(x) dx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin \left( \frac{x}{a} \right) \quad ( x  < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \arctan \left( \frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh} \left( \frac{x}{a} \right), \ln \{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh} \left( \frac{x}{a} \right), \ln \{x + \sqrt{x^2 + a^2}\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right  = \frac{1}{a} \operatorname{artanh} \left( \frac{x}{a} \right) \quad ( x  < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $

## Statistics

### Discrete distributions

For a discrete random variable  $X$  taking values  $x_i$  with probabilities  $P(X = x_i)$

Expectation (mean):  $E(X) = \mu = \sum x_i P(X = x_i)$

Variance:  $\text{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 P(X = x_i) = \sum x_i^2 P(X = x_i) - \mu^2$

### Discrete distributions

Standard discrete distributions:

Distribution of $X$	$P(X = x)$	Mean	Variance
Binomial $B(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	$np$	$np(1-p)$
Poisson $Po(\lambda)$	$e^{-\lambda} \frac{\lambda^x}{x!}$	$\lambda$	$\lambda$

### Continuous distributions

For a continuous random variable  $X$  having probability density function  $f$

Expectation (mean):  $E(X) = \mu = \int x f(x) dx$

Variance:  $\text{Var}(X) = \sigma^2 = \int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$

For a function  $g(X)$ :  $E(g(X)) = \int g(x) f(x) dx$

Cumulative distribution function:  $F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(t) dt$

Standard continuous distribution:

Distribution of $X$	P.D.F.	Mean	Variance
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mu$	$\sigma^2$
Uniform (Rectangular) on $[a, b]$	$\frac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$

## Correlation and regression

For a set of  $n$  pairs of values  $(x_i, y_i)$

$$S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

$$S_{yy} = \Sigma(y_i - \bar{y})^2 = \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}$$

$$S_{xy} = \Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}$$

The product moment correlation coefficient is

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\{\Sigma(x_i - \bar{x})^2\}\{\Sigma(y_i - \bar{y})^2\}}} = \frac{\Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}}{\sqrt{\left(\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right)\left(\Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}\right)}}$$

The regression coefficient of  $y$  on  $x$  is  $b = \frac{S_{xy}}{S_{xx}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$

Least squares regression line of  $y$  on  $x$  is  $y = a + bx$  where  $a = \bar{y} - b\bar{x}$

$$\text{Residual Sum of Squares (RSS)} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} = S_{yy} (1 - r^2)$$

$$\text{Spearman's rank correlation coefficient is } r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

## Non-parametric tests

Goodness-of-fit test and contingency tables:  $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi_v^2$

## Statistical tables

The following statistical tables are required for AS Level Further Mathematics:

*Binomial Cumulative Distribution Function* (see page 29)

*Poisson Cumulative Distribution Function* (see page 35)

*Percentage Points of the  $\chi^2$  Distribution* (see page 36)

*Critical Values for Correlation Coefficients: Product Moment Coefficient and Spearman's Coefficient* (see page 37)

*Random Numbers* (see page 38)

## Mechanics

### Centres of mass

For uniform bodies:

Triangular lamina:  $\frac{2}{3}$  along median from vertex

Circular arc, radius  $r$ , angle at centre  $2\alpha$  :  $\frac{r \sin \alpha}{\alpha}$  from centre

Sector of circle, radius  $r$ , angle at centre  $2\alpha$  :  $\frac{2r \sin \alpha}{3\alpha}$  from centre

